

# NON-COHEN-MACAULAY CANONICAL SINGULARITIES

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*Dedicated to Lawrence Ein on the occasion of his 60th birthday*

## 1. INTRODUCTION

Since the first counter-example to Kodaira vanishing in positive characteristic was constructed by Raynaud [Ray78] many other counter-examples have been found satisfying various prescribed properties [DI87, Eke88, SB91, Kol96, Lau96, Muk13, dCF15, CT16a, CT16b]. An elementary counter-example for which the line bundle violating Kodaira vanishing is very ample was constructed by Lauritzen and Rao in [LR97]. Let us denote it by  $X$ . It is straightforward from the construction that  $X$  is a rational variety and for  $p = 2$  and  $\dim X = 6$  it is Fano. Let  $Z$  denote the cone over  $X$  using the embedding given by the global sections of the very ample line bundle violating Kodaira. It is well-known that a cone over a Fano variety has klt singularities if  $K_Z$  is  $\mathbb{Q}$ -Cartier. (See Definition 1.3.) The failure of Kodaira vanishing on  $X$  implies that  $Z$  will not have Cohen-Macaulay singularities, in particular it does not have rational singularities. As pointed out by Hélène Esnault, although in this example  $K_Z$  is not  $\mathbb{Q}$ -Cartier, one can easily find a boundary  $\Delta$  on  $Z$  that makes  $K_Z + \Delta$   $\mathbb{Q}$ -Cartier, and hence the pair  $(Z, \Delta)$  klt. In other words Lauritzen and Rao's counter-example to Kodaira vanishing produces a klt pair  $(Z, \Delta)$  such that  $Z$  is not Cohen-Macaulay. This provides a counter-example to Elkik's theorem [Elk81], [KM98, 5.22] in positive characteristic.

The purpose of this note is to show that one can use  $X$  to produce even more interesting singularities. I will demonstrate below that in fact the very ample line bundle  $\omega_X^{-2}$  also violates Kodaira vanishing and hence leads to a cone whose canonical sheaf is a line bundle, has canonical singularities, and is not Cohen-Macaulay. Of course, then it also does not have rational singularities. In other words, the purpose of this note is to prove the following.

**Theorem 1.1.** *Let  $k$  be a field of characteristic 2. Then there exists a Fano variety  $X$  over  $k$  such that*

- (i)  $\dim X = 6$ ,
- (ii)  $\omega_X^{-1}$  is very ample, and
- (iii)  $\omega_X^{-2}$  violates Kodaira vanishing.

One might ask if there is a similar example in smaller dimensions. It follows from [CT16b, A.1] that there are no Del Pezzo surfaces with this property. Hence a similar example in smaller dimension would be at least 3-dimensional. One might also ask if there is a similar example where  $\omega_X^{-1}$  violates Kodaira vanishing. The example here is certainly not such and it is well-known that no such example exists for  $\dim X = 2, 3$  [SB97, Sch07, Mad16]. While this is an interesting question, it happens to be irrelevant for the purposes of the present

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article. The more interesting question is whether there are similar examples in all positive characteristics.

My main interest in the above result lies in the following application. By taking the cone over  $X$  given by the embedding induced by the global sections of  $\omega_X^{-1}$  we obtain the following.

**Theorem 1.2.** *Let  $k$  be a field of characteristic 2. Then there exists a variety  $Z$  over  $k$  with the following properties:*

- (a)  $\dim Z = 7$ ,
- (b)  $Z$  has a single isolated canonical singularity, and admits a resolution of singularities by a smooth variety over  $k$ ,
- (c)  $\omega_Z$  is a line bundle,
- (d)  $Z$  is not Cohen-Macaulay, in particular,  $Z$  is not Gorenstein and does not have rational singularities.

Again, one might ask if there are such singularities in smaller dimensions. Of course, if one finds examples such as in Theorem 1.1 in smaller dimensions, that would provide smaller dimensional examples for Theorem 1.2 as well. However, as mentioned above there are no examples similar to Theorem 1.1 in dimension 2 which makes it an interesting question whether there exist 3-dimensional canonical singularities, perhaps even of index 1, that are not Cohen-Macaulay. Note that in characteristic 2 there exist 3-dimensional non-Cohen-Macaulay klt singularities by [CT16a, Thm 1.3]. And again, the possibly more interesting question is whether there are such singularities in all positive characteristics.

**DEFINITION 1.3.** Let  $X$  be a smooth projective variety over  $k$  and  $\mathcal{L}$  an ample line bundle on  $X$ . Then we will say that  $\mathcal{L}$  *violates Kodaira vanishing* if there exists an  $i < \dim X$  such that  $H^i(X, \mathcal{L}^{-1}) \neq 0$ . This is equivalent to that  $H^{\dim X - i}(X, \mathcal{L} \otimes \omega_X) \neq 0$  by Serre duality.

The canonical divisor of a normal variety  $Z$  is denoted, as usual, by  $K_Z$  and the associated reflexive sheaf of rank 1, the canonical sheaf, is denoted by  $\omega_Z$ . I.e.,  $\omega_Z \simeq \mathcal{O}_Z(K_Z)$ . A Weil divisor  $D$  on  $Z$  is a  $\mathbb{Q}$ -Cartier divisor if there exists a non-zero  $m \in \mathbb{N}$  such that  $mD$  is Cartier. A normal variety  $Z$  is said to have *rational singularities* if for a resolution of singularities  $\phi : \tilde{Z} \rightarrow Z$  the following conditions hold:

- (i)  $\mathcal{R}^i \phi_* \mathcal{O}_{\tilde{Z}} = 0$  for  $i > 0$ , and
- (ii)  $\mathcal{R}^i f_* \omega_{\tilde{Z}} = 0$  for  $i > 0$ .

Note that in characteristic 0 the condition in (ii) is automatic by the Grauert-Riemenschneider vanishing theorem [GR70], [KM98, 2.68].

For the definition of *klt* and *canonical* singularities the reader is referred to [Kol13, 2.8].

Rational singularities are Cohen-Macaulay by the following well-known lemma. A very short proof is included for the convenience of the reader. This shows that if  $Z$  is not Cohen-Macaulay, then for any resolution of singularities  $\phi : \tilde{Z} \rightarrow Z$ , there exists an  $i > 0$  such that either  $\mathcal{R}^i \phi_* \mathcal{O}_{\tilde{Z}} \neq 0$  or  $\mathcal{R}^i \phi_* \omega_{\tilde{Z}} \neq 0$ .

**Lemma 1.4.** *Let  $Z$  be a scheme with rational singularities. Then  $Z$  is Cohen-Macaulay.*

*Proof.* Let  $d = \dim Z$  and let  $\phi : \tilde{Z} \rightarrow Z$  be a resolution of singularities of  $Z$ . This implies that  $\mathcal{O}_Z \simeq \mathcal{R} \phi_* \mathcal{O}_{\tilde{Z}}$  and  $\omega_Z \simeq \mathcal{R} \phi_* \omega_{\tilde{Z}}$ . Then by Grothendieck duality

$$\omega_Z[d] \simeq \mathcal{R} \phi_* \omega_{\tilde{Z}}[d] \simeq \mathcal{R} \phi_* \mathcal{R} \mathcal{H}om_{\tilde{Z}}(\mathcal{O}_{\tilde{Z}}, \omega_{\tilde{Z}}^\bullet) \simeq \mathcal{R} \mathcal{H}om_Z(\mathcal{R} \phi_* \mathcal{O}_{\tilde{Z}}, \omega_Z^\bullet) \simeq \omega_Z^\bullet,$$

and hence  $Z$  is Cohen-Macaulay.  $\square$

## 2. NON-COHEN-MACAULAY SINGULARITIES VIA FAILURE OF KODAIRA VANISHING

First I will review the more-or-less well-known idea of constructing non-Cohen-Macaulay singularities as cones over varieties violating Kodaira vanishing.

(2.1) Let  $X$  be a smooth projective variety over a field  $k$  of characteristic  $p > 0$ ,  $\mathcal{L}$  a very ample line bundle on  $X$ , and  $Z$  the cone over  $X$  embedded via the global sections of  $\mathcal{L}$ .

Then we have the following well-known criterion cf. [Kol13, 3.11]:

(2.2)  $Z$  is Cohen-Macaulay if and only if  $H^i(X, \mathcal{L}^q) = 0$  for all  $0 < i < \dim X$  and  $q \in \mathbb{Z}$ .

This implies for example that cones over varieties whose structure sheaves have non-trivial *middle* cohomology, for instance abelian varieties of dimension at least 2, are not Cohen-Macaulay. It also implies that

(2.3) if some power of  $\mathcal{L}$  violates Kodaira vanishing, then  $Z$  is not Cohen-Macaulay.

Next recall that the canonical divisor of a canonical singularity is  $\mathbb{Q}$ -Cartier and observe that in the above construction

(2.4) If  $\omega_X^r \simeq \mathcal{L}^q$  for some  $r, q \in \mathbb{Z}$ ,  $r \neq 0$ , then  $K_Z$  is  $\mathbb{Q}$ -Cartier of index at most  $r$ .

Note that even if this last condition fails,  $Z$  may still provide an example of a klt singularity with an appropriate boundary. Since  $\mathcal{L}$  is ample, there exist  $r, q \in \mathbb{N}$ ,  $r, q > 0$  such that  $\mathcal{N} = \mathcal{L}^q \otimes \omega_X^{-r}$  is an ample line bundle. Let  $N$  be a general member of the complete linear system corresponding to  $\mathcal{N}^m$  for some  $m \gg 0$ ,  $\hat{N} \subseteq Z$  the cone over  $N$ , and  $\Delta := \frac{1}{rm} \hat{N}$ . Then  $\omega_Z(\Delta)^{[rm]}$  is a line bundle on  $Z$  and hence  $K_Z + \Delta$  is  $\mathbb{Q}$ -Cartier.

Next, recall that blowing up the cone point of  $Z$  provides a resolution of singularities  $\phi : \tilde{Z} \rightarrow Z$  with exceptional divisor  $E \simeq X$ . Then one has that

$$K_{\tilde{Z}} + \phi_*^{-1} \Delta \sim_{\mathbb{Q}} \phi^*(K_Z + \Delta) + aE$$

for some  $a \in \mathbb{Q}$ . By the adjunction formula this implies that

$$K_E \sim_{\mathbb{Q}} ((a+1)E - \phi_*^{-1} \Delta)|_E.$$

It follows that if  $X$  is Fano, then one must have  $a > -1$  and hence  $(Z, \Delta)$  is klt (cf. [Kol13, 3.1]).

Observe that in case (2.4) holds, and one replaces  $N$  with  $\emptyset$ , then one obtains that  $(Z, 0)$  is klt. In particular,

(2.5) if (2.4) holds with  $r = 1$ , then  $K_Z$  is Cartier and  $Z$  has only canonical singularities.

Let me summarize what we found out in this section:

**Proposition 2.6.** *In addition to the definitions in (2.1) assume that  $X$  is Fano and some power of  $\mathcal{L}$  violates Kodaira vanishing. Then there exists a  $\mathbb{Q}$ -divisor  $\Delta$  on  $Z$  such that*

- (i)  $(Z, \Delta)$  has klt singularities,
- (ii)  $Z$  is not Cohen-Macaulay, and hence in particular has non-rational singularities, and
- (iii) if  $\omega_X \simeq \mathcal{L}^q$  for some  $q \in \mathbb{Z}$ , then  $Z$  has canonical singularities.

Note that this fact has been observed by several people including Hélène Esnault and János Kollár (cf. [Kol13, 3.1]).

### 3. THE CONSTRUCTION OF LAURITZEN AND RAO

Next, I will recall the construction of Lauritzen and Rao from [LR97].

Let  $V$  be a vector space of dimension  $n + 1$  over a field  $k$  of characteristic  $p$  where  $p \geq n - 1 \geq 2$ , and let  $\mathbb{P}(V) \simeq \mathbb{P}^n$  be the associated projective space of dimension  $n$ . Let  $W := \mathbb{P}(V) \times \mathbb{P}(V^\vee)$  and for  $a, b \in \mathbb{Z}$  let  $\mathcal{O}_W(a, b)$  denote the line bundle  $\mathcal{O}_{\mathbb{P}(V)}(a) \boxtimes \mathcal{O}_{\mathbb{P}(V^\vee)}(b)$  on  $W$ . Next let  $\mathcal{A}$  be the locally free sheaf defined by the short exact sequence

$$(3.1) \quad 0 \longrightarrow \mathcal{A} \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(1) \longrightarrow 0,$$

and let  $\alpha : Y := \mathbb{P}(\mathcal{A}^\vee) \rightarrow \mathbb{P}(V)$  be the projective space bundle over  $\mathbb{P}(V)$  associated to  $\mathcal{A}^\vee$ . Let  $\mathcal{O}_\alpha(1)$  denote the corresponding tautological line bundle on  $Y$ . Then there exists another associated short exact sequence on  $Y$ :

$$(3.2) \quad 0 \longrightarrow \mathcal{G} \longrightarrow \alpha^* \mathcal{A}^\vee \longrightarrow \mathcal{O}_\alpha(1) \longrightarrow 0,$$

which defines the locally free sheaf  $\mathcal{G}$  on  $Y$ . It is shown in [LR97, p.23] that  $Y$  admits a closed embedding into  $W \simeq \mathbb{P}^n \times \mathbb{P}^n$  with bihomogenous coordinate ring

$$(3.3) \quad k[x_0, \dots, x_n; y_0, \dots, y_n] / (\sum x_i y_i).$$

In particular, the ideal sheaf of  $Y$  in  $W$  is  $\mathcal{O}_W(-1, -1)$ . Let  $\mathcal{O}_Y(a, b) := \mathcal{O}_W(a, b)|_Y$ . Then it follows easily that

$$(3.4) \quad \omega_Y \simeq \mathcal{O}_Y(-n, -n), \quad \alpha^* \mathcal{O}_{\mathbb{P}(V)}(a) \simeq \mathcal{O}_Y(a, 0), \quad \text{and} \quad \mathcal{O}_\alpha(b) \simeq \mathcal{O}_Y(0, b)$$

Let  $\eta$  be defined as the composition of the natural morphisms induced by the morphisms in (3.1) and (3.2) using the isomorphisms in (3.4):

$$(3.5) \quad \begin{array}{c} \xrightarrow{\eta} \\ V^\vee \otimes \mathcal{O}_Y \longrightarrow \alpha^* \mathcal{A}^\vee \longrightarrow \mathcal{O}_Y(0, 1) \end{array}$$

Then we have the following commutative diagram, where  $\mathcal{B} = \ker \eta$ :

$$(3.6) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_Y(-1, 0) & \xrightarrow{=} & \mathcal{O}_Y(-1, 0) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{B} & \longrightarrow & V^\vee \otimes \mathcal{O}_Y & \xrightarrow{\eta} & \mathcal{O}_Y(0, 1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \alpha^* \mathcal{A}^\vee & \longrightarrow & \mathcal{O}_Y(0, 1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Finally, let  $\mathcal{G}' = \mathcal{G} \otimes \mathcal{O}_\alpha(1)$  and  $\pi : X := \mathbb{P}(F^* \mathcal{G}') \rightarrow Y$ , where  $F : Y \rightarrow Y$  is the absolute Frobenius morphism of  $Y$ . Note that by construction  $\dim X = 3n - 3$  and  $\dim Y = 2n - 1$ .

(3.7) Again, it is shown in [LR97, p.23] that the tautological line bundle of  $\pi$ , denoted by  $\mathcal{O}_\pi(1)$ , is globally generated and the line bundle  $\pi^*\mathcal{O}_Y(1, 1) \otimes \mathcal{O}_\pi(1)$  is very ample. It follows that  $\pi^*\mathcal{O}_Y(1, 1) \otimes \mathcal{O}_\pi(q)$  is also very ample for any  $q > 0$ .

Using the formula for the canonical bundle of a projective space bundle, one obtains that

$$(3.8) \quad \omega_X \simeq \pi^*\mathcal{O}_Y(p - n, p(n - 2) - n) \otimes \mathcal{O}_\pi(-n + 1)$$

As it was pointed out by Hélène Esnault if one chooses the values  $p = 2$  and  $n = 3$ , then  $X$  is a Fano variety and hence there exists a klt pair  $(Z, \Delta)$  where  $Z$  is not Cohen-Macaulay, in particular, it does not have rational singularities cf. [Proposition 2.6](#).

#### 4. A FANO VARIETY VIOLATING KODAIRA VANISHING

We will use the above construction and prove that if  $p = 2$  and  $n = 3$ , then the very ample line bundle  $\omega_X^{-2}$  violates Kodaira vanishing. To do this, first we need to compute a few auxiliary cohomology groups. We will keep using the notation introduced in [Section 3](#).

**Lemma 4.1.** *Let  $a, b \in \mathbb{Z}$ . Then*

$$H^i(Y, \mathcal{O}_Y(a, b)) = 0$$

*if either  $a$  and  $b$  are arbitrary and  $0 < i < n - 1$ , or  $a, b > -n$  and  $i > 0$ , or at least one of  $a$  and  $b$  is negative and  $i = 0$ .*

*Proof.* By (3.3) we have the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_W(a - 1, b - 1) \longrightarrow \mathcal{O}_W(a, b) \longrightarrow \mathcal{O}_Y(a, b) \longrightarrow 0$$

Since  $W \simeq \mathbb{P}^n \times \mathbb{P}^n$ , the first two (non-zero) sheaves above have no cohomology for  $a, b > -n$  and  $i > 0$  and hence the same is true for the third one. Then the case of  $i = 0$  follows from the Künneth formula.  $\square$

**Lemma 4.2.** *Let  $a, b \in \mathbb{Z}$ . Then*

$$\begin{aligned} H^1(Y, \mathcal{O}_Y(a, b) \otimes F^*\mathcal{B}) &\simeq \\ &\simeq \text{coker} [H^0(Y, \mathcal{O}_Y(a, b) \otimes F^*(V^\vee \otimes \mathcal{O}_Y)) \xrightarrow{\eta_1 := F^*\eta} H^0(Y, \mathcal{O}_Y(a, b + p))] \end{aligned}$$

*where  $\eta_1 = F^*\eta$  is induced by the morphism  $\eta$  defined in (3.5). In particular, if either  $a < 0$  or  $b < -p$ , then*

$$H^1(Y, \mathcal{O}_Y(a, b) \otimes F^*\mathcal{B}) = 0.$$

*Proof.* Consider the Frobenius pull-back of the middle row of the diagram in (3.6) twisted with  $\mathcal{O}_Y(a, b)$ :

$$0 \longrightarrow \mathcal{O}_Y(a, b) \otimes F^*\mathcal{B} \longrightarrow \mathcal{O}_Y(a, b) \otimes F^*(V^\vee \otimes \mathcal{O}_Y) \longrightarrow \mathcal{O}_Y(a, b + p) \longrightarrow 0.$$

Then, since  $n > 2$ , both statements follow from [Lemma 4.1](#).  $\square$

**Lemma 4.3.** *Let  $a, b \in \mathbb{Z}$ . Then the morphism induced by the (obvious) one in (3.6) is an isomorphism:*

$$H^1(\mathcal{O}_Y(a, b) \otimes F^*\mathcal{B}) \xrightarrow{\simeq} H^1(\mathcal{O}_Y(a, b) \otimes F^*\mathcal{G}).$$

Furthermore, if  $a < p$  or  $b < -p$ , then the natural morphism induced by the same morphism as above is an injection:

$$H^1(\mathcal{O}_Y(a, b) \otimes \mathrm{Sym}^2 F^* \mathcal{B}) \hookrightarrow H^1(\mathcal{O}_Y(a, b) \otimes \mathrm{Sym}^2 F^* \mathcal{G}).$$

*Proof.* Consider the Frobenius pull-back of the first vertical short exact sequence from (3.6):

$$0 \longrightarrow \mathcal{O}_Y(-p, 0) \longrightarrow F^* \mathcal{B} \longrightarrow F^* \mathcal{G} \longrightarrow 0$$

Since  $n > 2$ , this, combined with Lemma 4.1, implies the first statement.

Next, observe that this short exact sequence also implies that there exists a filtration

$$\mathrm{Sym}^2 F^* \mathcal{B} \supseteq \mathcal{E} \supseteq \mathcal{O}_Y(-2p, 0)$$

such that (after twisting by  $\mathcal{O}_Y(a, b)$ ) we have the short exact sequences

$$(4.3.1) \quad 0 \longrightarrow \mathcal{O}_Y(a - 2p, b) \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathcal{E} \longrightarrow \mathcal{O}_Y(a - p, b) \otimes F^* \mathcal{G} \longrightarrow 0$$

and

$$(4.3.2) \quad 0 \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathcal{E} \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathrm{Sym}^2 F^* \mathcal{B} \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathrm{Sym}^2 F^* \mathcal{G} \longrightarrow 0$$

Then by the first statement, Lemma 4.2, Lemma 4.1, and (4.3.1) it follows that if  $a < p$  or  $b < -p$ , then

$$H^1(Y, \mathcal{O}_Y(a, b) \otimes \mathcal{E}) = 0.$$

By (4.3.2) this implies the second statement.  $\square$

**Lemma 4.4.** *Let  $a, b \in \mathbb{Z}$  such that  $a \geq 0$  and  $b > -n$ . Then*

$$H^1(Y, \mathcal{O}_Y(a, b) \otimes \mathrm{Sym}^2 F^* \mathcal{B}) \neq 0$$

*Proof.* Observe that the middle horizontal short exact sequence in the diagram (3.6) implies that there exists a filtration

$$\mathrm{Sym}^2 (F^*(V^\vee \otimes \mathcal{O}_Y)) \supseteq \mathcal{F} \supseteq \mathrm{Sym}^2 F^* \mathcal{B}$$

such that (after twisting by  $\mathcal{O}_Y(a, b)$ ) we have the short exact sequences

$$(4.4.1) \quad 0 \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathrm{Sym}^2 F^* \mathcal{B} \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathcal{F} \longrightarrow \mathcal{O}_Y(a, b + p) \otimes F^* \mathcal{B} \longrightarrow 0$$

and

$$(4.4.2) \quad 0 \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathcal{F} \longrightarrow \mathcal{O}_Y(a, b) \otimes \mathrm{Sym}^2 (F^*(V^\vee \otimes \mathcal{O}_Y)) \longrightarrow \mathcal{O}_Y(a, b + 2p) \longrightarrow 0.$$

Since  $n > 2$ , it follows from (4.4.2) and Lemma 4.1 that

$$H^2(Y, \mathcal{O}_Y(a, b) \otimes \mathcal{F}) = 0$$

and that

$$\begin{aligned} H^1(Y, \mathcal{O}_Y(a, b) \otimes \mathcal{F}) &\simeq \\ &\simeq \mathrm{coker} \left[ H^0(Y, \mathcal{O}_Y(a, b) \otimes \mathrm{Sym}^2 (F^*(V^\vee \otimes \mathcal{O}_Y))) \xrightarrow{\eta_2} H^0(Y, \mathcal{O}_Y(a, b + 2p)) \right]. \end{aligned}$$

Note that the morphism  $\eta_2$  here is given by the matrix  $[y_i^p y_j^p \mid i, j = 0, \dots, n]$ .

Furthermore, it follows from [Lemma 4.2](#) that

$$\begin{aligned} H^1(Y, \mathcal{O}_Y(a, b+p) \otimes F^* \mathcal{B}) &\simeq \\ &\simeq \text{coker} [H^0(Y, \mathcal{O}_Y(a, b+p) \otimes F^*(V^\vee \otimes \mathcal{O}_Y)) \xrightarrow{\eta_1} H^0(Y, \mathcal{O}_Y(a, b+2p))]. \end{aligned}$$

Note that the morphism  $\eta_1$  here is given by the matrix  $[y_i^p \mid i = 0, \dots, n]$ . Further note, that by assumption  $a \geq 0$  and  $b \geq -n+1 \geq -p$ , so  $H^0(Y, \mathcal{O}_Y(a, b+p) \otimes F^*(V^\vee \otimes \mathcal{O}_Y)) \neq 0$ . Then it is easy to see, for example from the description of  $\eta_1$  and  $\eta_2$  above, that

$$\text{im } \eta_2 \subsetneq \text{im } \eta_1,$$

and hence combined with [\(4.4.1\)](#) the above imply that

$$(4.4.3) \quad H^1(Y, \mathcal{O}_Y(a, b) \otimes \text{Sym}^2 F^* \mathcal{B}) \simeq \text{im } \eta_1 / \text{im } \eta_2 \neq 0. \quad \square$$

**REMARK 4.5.** Observe that the previous argument was the place where working in positive characteristic was crucial. The morphisms  $\eta_1$  and  $\eta_2$  are given by the  $p^{\text{th}}$  powers of the global sections of  $\mathcal{O}_Y(0, 1)$ . We obtain the non-trivial cokernels and the “gap” between them from the fact that the global sections of  $\mathcal{O}_Y(0, p)$  are not generated by these  $p^{\text{th}}$  powers.

This argument fails for several reasons in characteristic 0. First of all,  $p^{\text{th}}$  powers do not define an  $\mathcal{O}_Y$ -module homomorphism. Of course, they do not define one in any characteristic, which is the reason that we first have to pull-back everything by the Frobenius. However, the  $p^{\text{th}}$  powers do give an  $F^* \mathcal{O}_Y$ -module homomorphism. There is of course no Frobenius in characteristic 0, but one might think that then one could use another finite morphism to pull-back these sections and thereby replacing the global sections by an appropriate power. However, in characteristic 0 this would mean switching to an actual cover many of whose properties would change. For instance, very likely that cover would no longer be Fano or even have negative Kodaira dimension and other parts of the proof would break down.

To summarize, the reason this argument works in positive characteristic is that there is a high degree endomorphism which is one-to-one on points. Then again, this is not surprising at all as this is usually the reason when a statement holds in positive characteristic but not in characteristic 0.

**Theorem 4.6.** *If  $p \leq n = 3$ , then  $\dim X = 6$ , and  $H^5(X, \omega_X^2) \neq 0$ .*

*Proof.* By Serre duality and [\(3.8\)](#) we have that

$$\begin{aligned} (4.6.1) \quad H^i(X, \omega_X^2)^\vee &\simeq H^{6-i}(X, \omega_X^{-1}) \simeq H^{6-i}(X, \pi^* \mathcal{O}_Y(3-p, 3-p) \otimes \mathcal{O}_\pi(2)) \\ &\simeq H^{6-i}(Y, \mathcal{O}_Y(3-p, 3-p) \otimes \pi_* \mathcal{O}_\pi(2)) \\ &\simeq H^{6-i}(Y, \mathcal{O}_Y(3-p, 3-p) \otimes \text{Sym}^2 F^* \mathcal{G}') \\ &\simeq H^{6-i}(Y, \mathcal{O}_Y(3-p, 3+p) \otimes \text{Sym}^2 F^* \mathcal{G}) \end{aligned}$$

Since  $p > a = 3-p \geq$  and  $b = 3+p > -n = -3$ , the statement follows from [Lemma 4.3](#) and [Lemma 4.4](#).  $\square$

This might seem to give a desired example in  $p = 3$  as well, but this non-vanishing is only interesting when  $X$  is Fano, i.e., when  $\omega_X^{-1}$  is ample and that only holds when  $p = 2$ .

**Corollary 4.7.** *If  $n = 3$  and  $p = 2$ , then  $X$  is a Fano variety on which  $\omega_X^{-1}$  is very ample and  $\omega_X^{-2}$  violates Kodaira vanishing. In particular, [Theorem 1.1](#) follows.*



*Proof.* If  $n = 3$  and  $p = 2$ , then by (3.8)  $\omega_X \simeq \pi^* \mathcal{O}_Y(-1, -1) \otimes \mathcal{O}_\pi(-2)$  and hence  $\omega_X^{-1}$  is very ample by (3.7). By Theorem 4.6,  $\omega_X^{-2}$  violates Kodaira vanishing.  $\square$

**Corollary 4.8.** *Theorem 1.2 holds.*

*Proof.* Let  $Z$  be the cone over  $X$  embedded by the global sections of the very ample line bundle  $\omega_X^{-1}$ . Then the statement follows from Corollary 4.7 and Proposition 2.6.  $\square$

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